Melnikov function and homoclinic chaos induced by weak perturbations

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The effect of noise on the possible occurrence of chaos in systems with a homoclinic orbit (e.g., the Duffing equation) was recently considered by Bulsara, Schieve, and Jacobs [Phys. Rev. A **41**, 668 (1990)], and Schieve and Bulsara [Phys. Rev. A **41**, 1172 (1990)], who adopted an approach based on a redefinition of the Melnikov function. We show that this redefinition is unsatisfactory and leads to incorrect results.

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Consider the two-dimensional conservation system

$$\ddot{x}+f(x)=0\ , \tag{1}$$

where $f(x)$ is a nonlinear function of the dynamic variable $x(t)$ such that Eq. (1) possesses a homoclinic trajectory $x_s(t)$. We now introduce on the right-hand side of Eq. (1) the perturbation $\epsilon[\gamma\cos(\omega t)-k\dot{x}(t)]$, $\epsilon>0$. For ϵ sufficiently small, the saddle point of Eq. (1) perturbs to a nearby hyperbolic invariant manifold. The Melnikov function is related to the separation between the stable and unstable manifolds (e.g., Arrowsmith [1], p. 172), and has the expression

$$M(t_0,\theta_0)=\int_{-\infty}^{\infty}\dot{x}_s(t)[\gamma\cos(\omega t+\omega t_0+\theta_0)-k\dot{x}_s(t)]dt \tag{2}$$

(Wiggins [2], p. 507). Note that in the expression for the Melnikov function second-order terms are *not* taken into account. A clear explanation of the geometric meaning of t_0 and θ_0 is available in Wiggins [2], p. 487. For the purposes of this Comment it is convenient to keep constant the time t_0 which defines the particular orbit being considered and vary the angle $\theta_0=\omega\tau$ (where τ is a running parameter with the dimension of time) at which the Poincaré section is carried out, see Wiggins [2], p. 507. For additional details, see Simiu, Frey, and Grigoriu [3].

We now consider the perturbation

$$\epsilon[\gamma\cos(\omega t)-k\dot{x}(t)]+F(t)\ , \tag{3}$$

where $F(t)=\epsilon\gamma_1\cos(\omega_1 t)$. The Melnikov function then becomes

$$M(t_0,\theta_0,\theta_1)=M(t_0,\theta_0)+\int_{-\infty}^{\infty}\dot{x}_s(t)\gamma_1\cos(\omega_1 t+\omega_1 t_0+\theta_1)dt\ . \tag{4}$$

For any set of values t_0,θ_0,θ_1, the correction $\Delta M=M(t_0,\theta_0,\theta_1)-M(t_0,\theta_0)$, which accounts for the effect of the perturbation $F(t)$, involves only the parameters of Eq. (1) and those of the perturbing function $F(t)/\epsilon$. In particular, ΔM *does not depend upon the parameters* k, γ, *or* ω. Indeed, even if we introduce an infinite sum or Wiener integral [4] of terms sufficient to represent the paths of the Langevin noise considered in Ref. [5] or the white noise in Ref. [6], the correction ΔM remains functionally independent both of k and the parameters of the initial perturbation $\gamma\cos(\omega t)$. This is essentially because the Melnikov function is a linear functional of the perturbation, and follows from the Melnikov function's role as the first term in the perturbative expansion for the Melnikov distance.

We now describe the approach proposed by Bulsara, Schieve, and Jacobs [5]. A small Langevin perturbation $F(t)$ is introduced in Eq. (1), so that

$$\ddot{x}(t)+f(x)=F(t)\ . \tag{5}$$

According to Ref. [5], the introduction of this perturbation causes the original, unperturbed separatrix $x_s(t)$ to shift, giving

$$x(t)=x_s(t)+\delta x(t) \tag{6}$$

as the separatrix of the perturbed system (5). Given (6), the Melnikov function is then redefined by analogy with Eq. (2), and written (in a slightly different notation) as follows:

$$M_b=\int_{-\infty}^{\infty}\dot{x}(t)[\gamma\cos(\omega t+\theta_0)-k\dot{x}(t)]dt\ . \tag{7}$$

Equation (7) is obtained by, in effect, considering the system perturbed by $F(t)$ as a nominally unperturbed system, and applying to it the formalism used in the original Melnikov approach to the unperturbed system having the homoclinic trajectory $x_s(t)$.

Neglecting higher-order terms, the correction ΔM_b to the redefined Melnikov function is then obtained from Eqs. (7), (6), and (2) as follows:

$$\Delta M_b=-2k\int_{-\infty}^{\infty}\dot{x}_s(t)\delta x(t)dt+\gamma\int_{-\infty}^{\infty}[\cos(\omega t+\theta_0)]\delta\dot{x}(t)dt-k\int_{-\infty}^{\infty}\delta\dot{x}^2(t)dt\ . \tag{8}$$

We observe that the correction ΔM_b to the Melnikov function using the approach of Bulsara *et al.* depends on the parameter k and on the parameters γ and ω, of the initial perturbation $\gamma\cos(\omega t)$. This unsatisfactory situation directly contradicts the observation made above that the change in the Melnikov function depends on neither k nor the initial perturbation $\gamma\cos(\omega t)$. The redefined Melnikov function M_b is not a linear functional of the perturbative forces present in the system and cannot serve as a first-order approximation to the Melnikov distance.

If it were true that the first-order term in the expression for the redefined Melnikov function M_b vanished on average, then the second-order terms would, on the average, indicate the possibility of chaos. For consistency, though, the second-order terms would have to include not only those appearing in M_b but all second-order terms contained in the perturbative expansion of the Melnikov distance. This is not done by Bulsara *et al.* This is a moot point, however, since as is made clear in the following example, on the average the first-order term in the Melnikov function does in fact not vanish.

To conclude, it is unjustified to use simple zeros of M_b, as in the standard Melnikov analysis, to mark on the average the transverse intersections of the stable and unstable manifolds of the hyperbolic saddle point.

Example. We consider for definiteness the case of the Duffing oscillator $[f(x)=-x+x^3]$ with the perturbation given by Eq. (3), where $F(t)/\epsilon=\gamma_1\cos\omega_1 t$. We assume that $\gamma>0$, $\gamma_1\neq 0$ (we do not impose the restriction $\gamma_1>0$), and that ω,ω_1 are incommensurate. Following Wiggins [3] (p. 463) and Wiggins [7] (p. 516), the necessary condition for the occurrence of chaos is

$$-4k/3+\gamma S(\omega)+|\gamma_1|S(\omega_1)>0 \ , \tag{9}$$

where

$$S(\omega)=\sqrt{2}\pi\omega\,\mathrm{sech}(\pi\omega/2) \ . \tag{10}$$

It is clear that for any given parameters k and ω, the presence of the perturbation $F(t)$ *lowers* the minimum (threshold) value of γ for which the occurrence of chaos is possible. In the absence of the perturbation $F(t)$, the Melnikov criterion is $-4k/3+\gamma S(\omega)>0$. Thus, the addition of the perturbation $F(t)$ represented by the term $|\gamma_1|S(\omega_1)$ in (9) lowers the minimum (threshold) value of γ for which chaos is possible. If, instead of confining ourselves to one amplitude γ_1, we consider an ensemble of values γ_1 with zero mean, then *the average effect of the perturbation $F(t)$ is still to lower the threshold for γ.* This is because $E[|\gamma_1|]>0$. Note that with γ_1 chosen to have zero mean, $F(t)$ is a zero mean process—its spectrum has all mass at angular frequency ω_1. We could introduce more harmonic perturbations sufficient to represent Langevin noise. Additional perturbative terms can only on average further lower the threshold. Thus, our argument does not depend on whether $F(t)$ is a harmonic function, a sum of harmonic functions, or, as in Bulsara *et al.*, weak Langevin noise. In general, noise cannot on average suppress homoclinic chaos in near-integrable systems. By contrast, the approach of Bulsara *et al.* leads to the result that "the presence of weak Langevin noise in a dissipative system suppresses, in the mean, homoclinic behavior that might normally be observed in the noise-free system" [5].

Numerical simulations. We performed numerical simulations to verify that noise does not on the average suppress homoclinic behavior in near-integrable systems. We considered a harmonically forced Duffing equation with parameters corresponding to the threshold case where the Melnikov distance in the system without noise is zero. In our simulations $\epsilon k=0.1$, $\epsilon\omega=1$, and $\epsilon\gamma=(4k/3)/[\sqrt{2}\pi\omega\,\mathrm{sech}(\pi\omega/2)]=0.075\,301\,81$, so that $-(4/3)k+\gamma S(\omega)=0$. Noise added to the excitation was simulated by using the stochastic process representation

$$F(t)=\epsilon\sigma(2/N)^{1/2}\sum_{k=1}^{N}\cos(\omega_k t+\phi_{k0}) \tag{11}$$

(Shinozuka [8]), where the phase angles ϕ_{k0} are uniformly distributed between 0 and 2π, the circular frequencies ω_k have probability density $p(\Omega)=g(\Omega)/(\epsilon\sigma)^2$, and $g(\Omega)$ is the power spectral density of $F(t)$. From the form of Eq. (11), and in view of the results obtained by Wiggins [7] (p. 467), we expect any realization of the stochastic perturbation to cause transverse intersections of the stable and unstable manifolds. We verified numerically that this is indeed the case. To allow replication of our results we reproduce, as an example, the parameters for one realization of the stochastic process $F(t)$ defined by Eq. (11), with $\epsilon\sigma=0.02$, $N=15$ [i.e., the common amplitude of the harmonics in Eq. (11) is $0.02(2/15)^{1/2}=0.007\,302\,97$], and a band-limited spectrum with constant ordinate in the interval $0<\Omega<2\pi$ and zero ordinate elsewhere:

$$\{\omega_1,\omega_2,\ldots,\omega_{15}\}=\{0.2177,0.6147,0.9834,1.3966,1.8073,2.1843,2.6103,2.8328,\ 3.4128,3.8018,4.1888,4.5886,5.006,5.3853,6.1843\}\ ,$$

$$\{\phi_{10},\phi_{20},\ldots,\phi_{150}\}=\{3.0473,2.5509,5.0328,3.9521,0.7979,3.1792,6.1952,3.4808,\ 2.5195,3.3489,1.6888,3.3552,1.7216,3.7384,2.0985\}\ .$$

We follow in our construction the procedure described by Wiggins [9] (p. 180). We portray lobe boundaries of portions of the intersecting stable and unstable manifolds at time $t=5\times 2\pi/\omega=10\pi$. This corresponds to a phase slice $\chi(\phi_{10}+5\times 2\pi\omega_1/\omega,\phi_{20}+5\times 2\pi\omega_2/\omega,\ldots,\phi_{150}+5\times 2\pi\omega_{15}/\omega)$ of the Poincaré map P_ϵ generated by the system's autonomous counterpart and the global cross section $\Sigma^{\phi_0}=\{(x_1,x_2;\phi_1,\phi_2,\ldots,\phi_{15})|\phi=\phi_0\}$ through the autonomous phase space $\{x_1, x_2; \phi_1, \phi_2, \ldots, \phi_{15}, \phi\}$ (Wiggins [9]). (Note that $\phi = \omega t + \phi_0$ and $\phi_0 = 0$.) Since the excitation is characterized by $N+1$ frequencies, P_ϵ possesses an N-dimensional normally hyperbolic invariant torus τ_ϵ which has $(N+1)$-dimensional stable and unstable local manifolds denoted $W^s_{\rm loc}(\tau_\epsilon)$ and $W^u_{\rm loc}(\tau_\epsilon)$, respectively. The global stable and unstable manifolds are defined as

$$W^s(\tau_\epsilon) = \bigcup_{n=0}^{\infty} P_\epsilon^{-n}[W^s_{\rm loc}(\tau_\epsilon)] ,$$

$$W^u(\tau_\epsilon) = \bigcup_{n=0}^{\infty} P_\epsilon^{n}[W^u_{\rm loc}(\tau_\epsilon)] .$$

The torus τ_ϵ intersects a plane $\chi(\phi_1, \phi_2, \ldots, \phi_{15}; \phi)$ in a unique point. In this example, for $\phi_i = \phi_{i0} (i = 1, 2, \ldots, 15)$ this point has coordinates $\{-0.024\,207\,850\,724\,634, 0.002\,384\,710\,615\,867\}$. The corresponding eigenvalues $\lambda_u = 0.950\,370\,901\,161\,105$, $\lambda_s = -1.050\,307\,901\,161\,105$ are obtained from the first variation of the perturbed equations of motion. λ_u defines the slope of the local unstable manifold needed to construct the curves representing the intersection of part of the global unstable manifold with the time slice for $t = 10\pi$. For $\phi_i = \phi_{i0} + 10 \times 2\pi\omega_i/\omega$ $(i = 1, 2, \ldots, 15)$, the coordinates of the point of intersection of τ_ϵ with the corresponding plane χ are $\{-0.024\,509\,650\,269\,781, 0.003\,145\,101\,332\,251\}$ and the eigenvalues are $\lambda_u = 0.950\,348\,854\,715\,673$, $\lambda_s = -1.050\,348\,854\,715\,673$. We obtain the curves being sought by intersecting trajectories evolved in forward time from the local unstable manifold at time $t = 0$ to 10π, and in backward time from the local stable manifold at time $t = 20\pi$ to 10π (recall that in our case $\omega = 1$).

We show curves so constructed in Fig. 1. As expected, Fig. 1 exhibits intersections between curves belonging to the stable manifold and those belonging to the unstable manifold. It follows from the arguments developed earlier that transverse intersections will be caused in our system by any realization of a similar stochastic process, regardless of spectral bandwidth and number of components N (see also Ref. [10]).

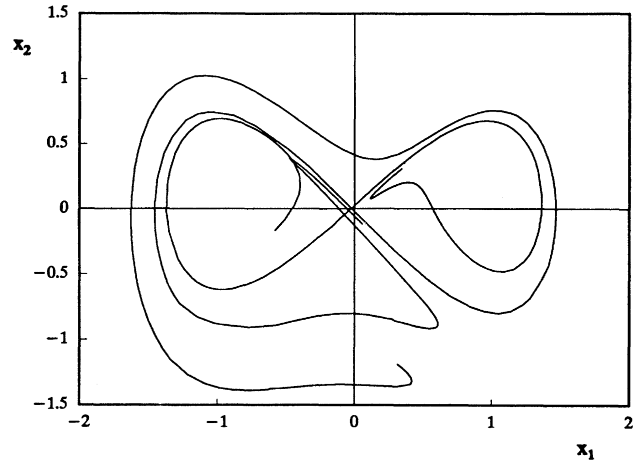

FIG. 1. Sample phase-space slice corresponding to a time $t = 10\pi$ for the Duffing equation with a stochastic perturbation given by Eq. (11).

Conclusions. First, as our example makes clear, the first-order effect of a noise perturbation $F(t)$ on the Melnikov function does not vanish on average, as asserted in Ref. [5]. Second, the redefined Melnikov function M_b introduced in Ref. [5] [Eq. (7)] omits the second-order terms inherent in the original derivation of the Melnikov distance. Thus, the second-order terms calculated from Eq. (7) would not reflect the totality of the second-order effects even if the redefined Melnikov function were mathematically meaningful. Third, and most important, the redefined Melnikov function M_b is not mathematically meaningful because it implies that the Melnikov distance may be referenced with respect to a set of separated manifolds. In fact, the total Melnikov distance should be referenced with respect to the homoclinic separatrix of the unperturbed equation.

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